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The thermal coherent state and its application to the one-dimensional field theory

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Abstract. We present a new formulation of the two-fluid model, the thermal coherent state for handling the relativistic quantum field theory at finite temperature.

Three models in one-dimensional space, the ϕ^4, sine-Gordon and the Schwinger model, are discussed.

1. Introduction

In two previous papers (Su *et al* 1983, Chen and Ni 1983), to be referred to hereafter as I and II, a method was developed for handling the temperature field theory in one-dimensional space. This method, based on the concept of the coherent state and treating the quantum fluctuations by the Green function approach, is essentially a two-fluid model as summarised in the last section of II. The coherent state, say ϕ_s, corresponds to the superfluid component and the incoherent part, say the quantised $\hat{\psi}$ around ϕ_s, to the normal fluid component. A perfect coherent state implies that it is at zero temperature. The incoherent thermal fluctuations increase with temperature and begin to destroy the long-range order, i.e. the phase correlation, until the latter totally vanishes at the critical temperature.

An alternative formulation, the thermal coherent state, will be presented in §2 which can be viewed as a mixed ensemble describing the coherent excitation on an incoherent background in thermal equilibrium. The concise representation and the formulae derived thereby enable us to treat more easily various systems where the effect of the coexistence of two components is concerned. We discuss three examples in one-dimensional space, the ϕ^4 model, sine-Gordon model and the Schwinger model in §§ 3, 4 and 5 respectively. Section 6 will be a summary and discussion. Three appendices give some mathematical details which are omitted in the text.

2. The thermal coherent state

It is known that the coherent state of a neutral scalar field is expressed as (I):

$$|f\rangle = N \exp\left(\sum_k f_k \hat{a}_k^+\right)|0\rangle$$
$$= N \exp\left(\int \mathrm{d}k\, f(k)\hat{a}^+(k)\right)|0\rangle \qquad (2.1)$$

with $\hat{a}_k^+ \leftrightarrow (2\pi/L)^{1/2}\hat{a}^+(k), f_k \leftrightarrow (2\pi/L)^{1/2}f(k)$ in one-dimensional space. The normalisation constant N is chosen so that $\langle f|f\rangle = 1$. The symbol $|0\rangle$ denotes the vacuum state, while $\hat{a}_k^+|0\rangle$ expresses the one-particle state with momentum k. It is important to mention that the mass of the particle, say μ, does not need to be fixed as an input mass parameter in the original Lagrangian, rather, it can be chosen at our disposal. For the relativistic field models we are dealing with in this paper, the most favourable choice is setting μ equal to the mass of quasiparticles. Being an independent or nearly independent elementary excitation of the whole system, the quasiparticle absorbs a considerable part of the mutual interactions between 'original' particles into its inner structure. Therefore the mass of the quasiparticle may be temperature dependent. The flexibility of choice of mass parameter for a relativistic field theory at finite temperature will be exhibited in the following discussion.

Now let us replace the vacuum state $|0\rangle$ in (2.1) by $|0\rangle_\beta$. For a boson system it is defined as (Donoghue and Holstein 1983):

$$_\beta\langle 0|\hat{a}_k^+\hat{a}_{k'}|0\rangle_\beta = \delta_{kk'} n_k, \qquad n_k = (e^{\beta E_k} - 1)^{-1} \tag{2.2}$$

where $E_k^2 = \mu^2 + k^2$ is the energy of the quasiparticle at temperature $T(=1/\beta)$. Equation (2.2) describes a stationary Bose-Einstein distribution of quasiparticles, the total number of which is not fixed so that there is no chemical potential in (2.2). Once μ has been chosen as the mass of the independent quasiparticle, i.e. of the free phonon, then as a good approximation, we have:

$$_\beta\langle 0|\hat{a}_p^+\hat{a}_q^+|0\rangle_\beta = {}_\beta\langle 0|\hat{a}_p\hat{a}_q|0\rangle_\beta = 0. \tag{2.3}$$

It is evident that the definition of $|0\rangle_\beta$ shown by the ensemble average (2.2) and (2.3) implies the existence of a heat bath in thermal equilibrium. The quasiparticles constitute an incoherent background on which we can construct further a thermal coherent state as

$$|f\rangle_\beta = N \exp\left(\sum_k f_k \hat{a}_k^+\right)|0\rangle_\beta \tag{2.4}$$

with

$$N^2 = \exp\left(-\sum_k |f_k|^2(1+n_k)\right) \tag{2.5}$$

for ensuring

$$_\beta\langle f|f\rangle_\beta = 1 \tag{2.6}$$

where the formula

$$e^A e^B = e^B e^A e^{[A,B]} \tag{2.7}$$

($[A, B]$ commutes with A and B) and equations (2.2) and (2.3) have been used. Under the same approximation, we can obtain the series of formulae:

$$_\beta\langle f|\hat{a}_p|f\rangle_\beta = f_p(1+n_p) \tag{2.8}$$

$$_\beta\langle f|\hat{a}_p\hat{a}_q|f\rangle_\beta = f_p f_q(1+n_p)(1+n_q) \tag{2.9}$$

$$_\beta\langle f|\hat{a}_p^+\hat{a}_q|f\rangle_\beta = \delta_{pq}n_p + f_p^* f_q(1+n_p)(1+n_q) \tag{2.10}$$

$$_\beta\langle f|\hat{a}_{p_1}\hat{a}_{p_2}\hat{a}_{p_3}\hat{a}_{p_4}|f\rangle_\beta = f_{p_1}f_{p_2}f_{p_3}f_{p_4}(1+n_{p_1})(1+n_{p_2})(1+n_{p_3})(1+n_{p_4}) \tag{2.11}$$

$$
\begin{aligned}
{}_\beta\langle f|\hat{a}^+_{p_1}a_{p_2}a_{p_3}a_{p_4}|f\rangle_\beta = & f^*_{p_1}f_{p_2}f_{p_3}f_{p_4}(1+n_{p_1})(1+n_{p_2})(1+n_{p_3})(1+n_{p_4}) \\
& + n_{p_1}[\delta_{p_1p_4}f_{p_2}f_{p_3}(1+n_{p_2})(1+n_{p_3}) \\
& + \delta_{p_1p_3}f_{p_2}f_{p_4}(1+n_{p_2})(1+n_{p_4}) + \delta_{p_1p_2}f_{p_3}f_{p_4}(1+n_{p_3})(1+n_{p_4})]
\end{aligned} \tag{2.12}
$$

$$
\begin{aligned}
{}_\beta\langle f|\hat{a}^+_{p_1}\hat{a}^+_{p_2}\hat{a}_{p_3}\hat{a}_{p_4}|f\rangle = & f^*_{p_1}f^*_{p_2}f_{p_3}f_{p_4}(1+n_{p_1})(1+n_{p_2})(1+n_{p_3})(1+n_{p_4}) \\
& + n_{p_1}[\delta_{p_1p_4}f^*_{p_2}f_{p_3}(1+n_{p_2})(1+n_{p_3}) + \delta_{p_1p_3}f^*_{p_2}f_{p_4}(1+n_{p_2})(1+n_{p_4})] \\
& + n_{p_2}[\delta_{p_2p_3}f^*_{p_1}f_{p_4}(1+n_{p_1})(1+n_{p_4}) + \delta_{p_2p_4}f^*_{p_1}f_{p_3}(1+n_{p_1})(1+n_{p_3})] \\
& + n_{p_1}n_{p_2}(\delta_{p_1p_3}\delta_{p_2p_4} + \delta_{p_1p_4}\delta_{p_2p_3}).
\end{aligned} \tag{2.13}
$$

We leave the proof of equation (2.10) to appendix 1. Then by use of the inductive method in mathematics, we are able to obtain formulae containing an arbitrary number of $\hat{a}^+_p$ and/or $\hat{a}_q$:

$$
\begin{aligned}
& {}_\beta\langle f|\hat{a}^+_{p_1}\ldots\hat{a}^+_{p_m}\hat{a}_{q_1}\ldots\hat{a}_{q_n}|f\rangle_\beta \\
& \quad = \prod_{i=1}^m f^*_{p_i}(1+n_{p_i})\prod_{j=1}^n f_{q_j}(1+n_{q_j}) \\
& \quad + \sum_{i=1}^m n_{p_i}\left(\sum_{j=1}^n \delta_{p_iq_j}\prod_{\substack{k\neq i\\ l\neq j}} f^*_{p_k}(1+n_{p_k})f_{q_l}(1+n_{q_l})\right) \\
& \quad + \sum_{\substack{i,k=1\\(i\neq k)}}^m n_{p_i}n_{p_k}\left(\sum_{j,k=1}^n \delta_{p_iq_j}\delta_{p_kq_l}\prod_{\substack{r\neq i,k\\ s\neq j,l}} f^*_{p_r}(1+n_{p_r})f_{q_s}(1+n_{q_s})\right) + \ldots.
\end{aligned} \tag{2.14}
$$

Obviously, the formulae will become even more formidable as m and n get larger. But we can derive an elegant formula for scalar field with infinite powers of self-interactions (see appendix 2):

$$
{}_\beta\langle f|:\mathrm{e}^{\mathrm{i}g\phi(x)}:|f\rangle_\beta = \exp(-K+G-G^*) \tag{2.15}
$$

where

$$
\phi(x) = \sum_k (2L\omega_k)^{-1/2}(\hat{a}_k\,\mathrm{e}^{\mathrm{i}kx} + \hat{a}^+_k\,\mathrm{e}^{-\mathrm{i}kx}) \tag{2.16}
$$

$$
K = g^2\sum_p n_p/2L\omega_p \tag{2.17}
$$

$$
G = G(x) = \mathrm{i}g\sum_p (2L\omega_p)^{-1/2}\,\mathrm{e}^{\mathrm{i}px}f_p(1+n_p). \tag{2.18}
$$

We shall discuss the use of these formulae in the following sections.

3. The ϕ^4 model in one-dimensional space

As in the first example, we revisit the real ϕ^4 model in one-dimensional space (see I). The Lagrangian density reads:

$$
\mathscr{L} = \tfrac{1}{2}(\partial\phi/\partial t)^2 - \tfrac{1}{2}(\partial\phi/\partial x)^2 + \tfrac{1}{4}m_b^2\phi^2 - \tfrac{1}{4}g^2\phi^4 \tag{3.1}
$$

while the normal ordered (i.e. renormalised) Hamiltonian is

$$H=\sum_k\left(\frac{1}{2\omega_k}(k^2+\omega_k^2-\tfrac{1}{2}m^2)\hat{a}_k^+\hat{a}_k+\frac{1}{4\omega_k}(k^2-\omega_k^2-\tfrac{1}{2}m^2)(\hat{a}_k^+\hat{a}_{-k}^++\hat{a}_k\hat{a}_{-k})\right)$$

$$+\frac{g^2}{16L}\sum_{k_1,k_2,k_3,k_4}\frac{\delta_{k_1+k_2+k_3+k_4,0}}{(\omega_{k_1}\omega_{k_2}\omega_{k_3}\omega_{k_4})^{1/2}}[\hat{a}_{k_1}\hat{a}_{k_2}\hat{a}_{k_3}\hat{a}_{k_4}+\hat{a}_{-k_1}^+\hat{a}_{-k_2}^+\hat{a}_{-k_3}^+\hat{a}_{-k_4}^+$$

$$+4(\hat{a}_{-k_1}^+\hat{a}_{k_2}\hat{a}_{k_3}\hat{a}_{k_4}+\hat{a}_{-k_1}^+\hat{a}_{-k_2}^+\hat{a}_{-k_3}^+\hat{a}_{k_4})+6\hat{a}_{-k_1}^+\hat{a}_{-k_2}^+\hat{a}_{k_3}\hat{a}_{k_4}], \tag{3.2}$$

with

$$m^2=m_b^2-(3g^2/\pi)\ln(2\Lambda/\mu). \tag{3.3}$$

Instead of using the method of the Bogoliubov transformation and introducing the temperature by the Green function approach, this time we evaluate the expectation value of the Hamiltonian in the thermal coherent directly with the result:

$$U={}_\beta\langle f|H|f\rangle_\beta=\frac{L}{4\pi}\int_{-\infty}^{\infty}\mathrm{d}k\,\frac{n_k}{\omega_k}\,(k^2+\omega_k^2-\tfrac{1}{2}m^2)+\frac{3g^2L}{16\pi^2}\left(\int_{-\infty}^{\infty}\mathrm{d}k\frac{n_k}{\omega_k}\right)^2$$

$$+\int_{-\infty}^{\infty}\mathrm{d}k\left(k^2-\tfrac{1}{2}m^2+\frac{3g^2}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}p\frac{n_p}{\omega_p}\right)\frac{f(k)f^*(k)(1+n_k)^2}{\omega_k} \tag{3.4}$$

$$+\frac{g^2}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\,\frac{f(k_1)(1+n_{k_1})}{\sqrt{\omega_{k_1}}}\,\frac{f(k_2)(1+n_{k_2})}{\sqrt{\omega_{k_2}}}$$

$$\times\frac{f(k_3)(1+n_{k_3})}{\sqrt{\omega_{k_3}}}\,\frac{f(-k_1-k_2-k_3)(1+n_{k_1+k_2+k_3})}{\sqrt{\omega_{k_1+k_2+k_3}}} \tag{3.5}$$

where the property of $f(-p)=f^*(p)$ has been used. Introducing

$$y(p)=(1/\sqrt{\omega_p})f(p)(1+n_p) \tag{3.6}$$

and making a Fourier transformation

$$y(p)=\int_{-\infty}^{\infty}\tilde{y}(x)\,\mathrm{e}^{-\mathrm{i}px}\,\mathrm{d}x, \tag{3.7}$$

then the variational condition

$$\delta U/\delta f(p)=0 \tag{3.8}$$

will lead to an equation for $\tilde{y}(x)$

$$(\mathrm{d}^2/\mathrm{d}x^2)\tilde{y}(x)+(\tfrac{1}{2}m^2-3g^2I_3/\pi)\tilde{y}(x)-4\pi g^2\tilde{y}^3(x)=0. \tag{3.9}$$

The definition of I_3 as well as I_1 and I_2 are listed in appendix 3. Thus we can get the following four kinds of solutions:

(a) $$\tilde{y}(x)=0,\qquad y(p)=0 \tag{3.10}$$

$$U(\mathrm{a})=(L/2\pi)[I_1(\beta\mu)T^2+I_2(\beta\mu)T^2-\tfrac{1}{2}m^2I_3+3g^2I_3^2/2\pi] \tag{3.11}$$

$$U(\mathrm{a})\xrightarrow[T\to 0]{}(L/2\pi)[(1+m/2T)T^2\,\mathrm{e}^{-m/T}+(3g^2/4m)T\,\mathrm{e}^{-2m/T}] \tag{3.12}$$

$$U(\mathrm{a})\xrightarrow[T\to\infty]{}L[(\tfrac{1}{6}\pi+3g^2/16m^2)T^2-\tfrac{1}{8}mT]. \tag{3.13}$$

At low temperature, this state is unstable against spontaneous symmetry breaking, i.e. against the phonon condensation to state (b) (see below). But as the temperature increases, looking at the ensemble average of $V''(\phi)$:

$$\langle V''(\phi)\rangle = \langle 3g^2\phi^2 - \tfrac{1}{2}m^2\rangle = (3g^2/\pi)I_3 - \tfrac{1}{2}m^2 + 12\pi g^2\langle\tilde{y}^2\rangle \tag{3.14}$$

in case (a),

$$\langle V''(\phi)\rangle|_{\mathrm{a}} = (3g^2/\pi)I_3 - \tfrac{1}{2}m^2. \tag{3.15}$$

We see that there exists a T_{c}, and $T > T_{\mathrm{c}}$, the expression (3.15) becomes positive, hence the state will be stable. One can choose (see I (6.18))

$$\mu^2 = \mu_a^2 = \langle V(\phi)\rangle|_{T \geqslant T_{\mathrm{c}}^{(\mathrm{a})}} \tag{3.16}$$

At the high-temperature limit (see I (6.13))

$$T_{\mathrm{c}}^{(\mathrm{a})} = (2m^3/9\surd 3g^2)\{1 - (9g^2/2\pi m^2)[\ln(g^2/m^2) - 0.0996]\}. \tag{3.17}$$

(b)
$$\tilde{y}(x) = \pm M/2(2\pi)^{1/2}g, \qquad y(p) = \pm(M/g)(\pi/2)^{1/2}\delta(p) \tag{3.18}$$

$$U(\mathrm{b}) = U(\mathrm{a}) - LM^4/16g^2 \tag{3.19}$$

where we choose

$$\mu^2 = \mu_{\mathrm{b}}^2 = \langle V''\rangle|_{\mathrm{b}} = M^2 = m^2 - 6g^2 I_3/\pi. \tag{3.20}$$

The condition of $M > 0$ leads to a critical temperature $T_{\mathrm{c}}^{(\mathrm{b})}$, which is the same as $T_{\mathrm{c}}^{(\mathrm{a})}$:

$$T_{\mathrm{c}}^{(\mathrm{b})} = T_{\mathrm{c}}^{(\mathrm{a})} = T_{\mathrm{c}}. \tag{3.21}$$

(c)
$$\tilde{y}(x) = \pm[1/2(2\pi)^{1/2}g]\, M \tanh \tfrac{1}{2}Mx \tag{3.22}$$

$$y(p) = \mp \mathrm{i}g^{-1}(\tfrac{1}{2}\pi)^{1/2} \operatorname{cosech} \pi p/M \tag{3.23}$$

$$U(\mathrm{c}) = U(\mathrm{b}) + M^3/3g^2. \tag{3.24}$$

The second term is just the mass of a soliton. In this case, since the condensation is not uniform in space, we can still use the expression (3.14) as the choice of μ^2 but with the symbol $\langle\ \rangle$ also implying the space average. Then

$$\mu_{\mathrm{c}}^2 = M^2 - \tfrac{3}{2}M^2\langle \operatorname{sech}^2 \tfrac{1}{2}Mx\rangle = M^2 - 6M/L \xrightarrow[L\to\infty]{} M^2. \tag{3.25}$$

The critical temperature is unchanged in the large-L limit

$$T_{\mathrm{c}}^{(\mathrm{c})} = T_{\mathrm{c}}. \tag{3.26}$$

(d) There is also a periodic solution expressed by the Jacobi elliptic function (Hammer 1981):

$$\tilde{y}(x) = \pm(M/2\sqrt{\pi}g)k(1+k^2)^{1/2}[\mathrm{sn}\, Mx[2(1+k^2)]]^{-1/2}. \tag{3.27}$$

The modulus k is determined by the boundary condition

$$ML/\surd 2(1+k^2)^{1/2} = 2nK(k) \tag{3.28}$$

where n, being an integer, may be interpreted as the number of soliton plus antisolitons in this system.

$$U(\mathrm{d}) = U(\mathrm{a}) - L\frac{k^2+2}{12(1+k^2)^2}\frac{M^4}{g^2} + n\frac{\surd 2E(k)}{(1+k^2)^{1/2}}\frac{M^3}{3g^2} \tag{3.29}$$

where

$$K(k)=\int_0^1 \frac{\mathrm{d}\xi}{(1-k^2\xi^2)^{1/2}(1-\xi^2)^{1/2}} \tag{3.30}$$

and

$$E(k)=\int_0^1 \left(\frac{1-k^2\xi^2}{1-\xi^2}\right)^{1/2} \mathrm{d}\xi \tag{3.31}$$

are the complete elliptic integrals of the first and second kind respectively. It is interesting to see that the cases (a), (b) and (c) can all be viewed as a special case of (d) with $k=0$; $n=0$ and $k=1$, $n=1$ respectively, noting that $\mathrm{sn}\, u|_{k=1}=\tanh u$.

The phonon mass in case (d) can also be chosen approximately as the space average of (3.14)

$$\begin{aligned}\mu_{\mathrm{d}}^2&=\frac{3g}{\pi} I_3-\tfrac{1}{2}m^2+3M^2\frac{k^2}{1+k^2}\left\langle \mathrm{sn}^2 \frac{Mx}{[2(1+k^2)]^{1/2}}\right\rangle\\ &=\frac{3g^2}{\pi} I_3-\tfrac{1}{2}m^2+\frac{3M^2}{1+k^2}\left(1-\frac{1}{2nK(k)}E(k)\right).\end{aligned} \tag{3.32}$$

Let us make a crude numerical analysis. Because $K(k)(E(k))$ is a montonic increasing (decreasing) function of k within the range $0\leqslant k\leqslant 1$, at a certain temperature the increase in number of solitons will lead to the decrease of k. There is a maximum value of n, say n_0 when $k=0$,

$$n_0=LM/\sqrt{2}\pi. \tag{3.33}$$

The quantity

$$\frac{U(\mathrm{d})-U(\mathrm{a})}{LM^4/g^2}=-\frac{(2+k^2)}{12(1+k^2)^2}+\frac{1}{6(1+k^2)}\frac{E(k)}{K(k)} \tag{3.34}$$

is not very sensitive to the change of k. It will arrive at the minimum (most negative) value when $k\to 1$. Therefore, the system will remain at the state with as few solitons as possible so long as it is allowed by the boundary condition.

4. The sine-Gordon system

Now we can use the thermal coherent state to discuss the sine-Gordon system with simplified calculations (see II).

$$\mathscr{L}=\tfrac{1}{2}(2\phi/\partial t)^2-\tfrac{1}{2}(\partial\phi/\partial x)^2+(m_{\mathrm{b}}^2/g^2)\cos g\phi-m_{\mathrm{b}}^2/g^2 \tag{4.1}$$

$$\mathscr{H}=N\mu[\tfrac{1}{2}\pi^2+\tfrac{1}{2}(\partial\phi/\partial x)^2-(m^2/g^2)\cos g\phi+D_0] \tag{4.2}$$

$$D_0=m_{\mathrm{b}}^2/g^2+(1/8\pi)\int \mathrm{d}k\,(2\omega_k-\mu^2/\omega_k)\;\cdot \tag{4.3}$$

$$m^2=m_{\mathrm{b}}^2\exp[(g^2/4\pi)\ln(\mu/2\Lambda)]. \tag{4.4}$$

Using equation (2.15) and noting $K=g^2I_3/2\pi$, we have

$${}_\beta\langle f|{:}\cos g\phi{:}|f\rangle_\beta=\mathrm{e}^{-K}\cos\tfrac{1}{2}g(\tilde{z}(x)+\tilde{z}^*(x))=\mathrm{e}^{-K}\cos g\tilde{z}(x) \tag{4.5}$$

where

$$\tilde{z}(x) = (1/\sqrt{\pi}) \int z(p)\, e^{ipx}\, dp \tag{4.6}$$

whereas

$$z(p) = \omega_p^{-1/2} f(p)(1+n_p). \tag{4.7}$$

The property of $f(p) = f^*(-p)$ has been used in the last step of equation (4.5).

Then it is easy to calculate the energy of the sine-Gordon system as

$$\begin{aligned} U &= \int {}_\beta\langle f|\mathscr{H}|f\rangle_\beta\, dx \\ &= L[(I_1+I_2)(T^2/2\pi)+D_0] + \tfrac{1}{2}\int dx\,(d\tilde{z}(x)/dx)^2 \\ &\quad -(m^2/g^2)\int dx\, e^{-K}\cos g\tilde{z}(x). \end{aligned} \tag{4.8}$$

We can derive from the condition

$$\delta U/\delta\tilde{z}(x) = 0 \tag{4.9}$$

that

$$d^2\tilde{z}(x)/dx^2 - (M^2/g)\sin g\tilde{z}(x) = 0 \tag{4.10}$$

where

$$M^2 = m^2\, e^{-K} = m^2 \exp(-g^2 I_3/2\pi). \tag{4.11}$$

Equation (4.10) is precisely the equation (3.16) in II†. There are three kinds of solutions:

(a)
$$\tilde{z}(x) = n\pi/g \qquad (n = 0, \pm1, \pm2, \dots) \tag{4.12}$$

$$U(a) = (L/2\pi)[(I_1+I_2)T^2 + 2\pi D_0] - (LM^2/g^2)(-)^n. \tag{4.13}$$

As before, the phonon mass μ is chosen as

$$\begin{aligned} \mu^2 &= \langle V''(\phi)\rangle \\ &= \langle m^2 \cos g\phi\rangle \\ &= M^2\langle \cos g\tilde{z}(x)\rangle. \end{aligned} \tag{4.14}$$

In case (a),

$$\mu_a^2 = (-)^n M^2, \tag{4.15}$$

so we see that n should be an even number.

(b)
$$\tilde{z}(x) = \pm(4/g)\tan^{-1} e^{\pm Mx}.$$

Noting that

$$\cos g\tilde{z}(x) = 1 - 2\,\text{sech}^2\, Mx \tag{4.17}$$

we can get the energy of the system as

$$U(b) = U(a) + 8M/g^2. \tag{4.18}$$

† Unfortunately, there is a double counting in contractions when evaluating the reduced Hamiltonian (3.9) in II. So (3.12) in II has to be corrected as (4.11) here. However the main scheme of II remains effective.

The second term is just the mass of one soliton (or antisoliton). The phonon mass square reads

$$\mu_b^2 = (M^2/L)\int_{-L/2}^{L/2} dx \cos g\tilde{z}(x) = M^2(1-4/ML) \xrightarrow[L\to\infty]{} M^2. \tag{4.19}$$

(c) There is also a multisoliton solution (Hammer and Shrauner 1984):

$$\tilde{z}(x) = (z/g)\cos^{-1}(k \text{ sn } Mx) \tag{4.20}$$

where the modulus k is determined by the boundary condition

$$ML = 2nK(k). \tag{4.21}$$

The energy of the system is

$$U(c) = (L/2\pi)[(I_1+I_2)T^2 + 2\pi D_0] + (LM^3/g^2)(2k^2-3) + 8nEM/g^2 \tag{4.22}$$

$$\frac{U(c)-U(a)}{LM^2/g^2} = -2(1-k^2) + 4(E(k)/K(k)). \tag{4.23}$$

Case (b) can be viewed as the special case of $k=1$ in (c) by noting the relation $\text{sn } u|_{K=1} = \tanh u$.

The mass of phonon can be chosen as (4.14)

$$\begin{aligned}\mu_c^2 &= M^2[2k^2 - 1 - 2k^2\langle \text{cn}^2 Mx\rangle] \\ &= M^2(1-2E/K) \xrightarrow[k\to 1]{} M^2.\end{aligned} \tag{4.24}$$

There is no critical temperature in any of these cases.

5. Schwinger model

The Schwinger model (Schwinger 1962) is defined as the quantum electrodynamics in one-dimensional space. The Lagrangian density reads

$$\mathscr{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - m)\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{5.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{5.2}$$

$$\gamma^0 = \sigma_3, \qquad \gamma^1 = i\sigma_2, \qquad \gamma^5 = \gamma^0\gamma^1 = \sigma_1, \qquad \bar{\psi} = \psi^+\gamma^0 \tag{5.3}$$

σ_i (i = 1, 2, 3) are Pauli matrices.

The Schwinger model has been investigated by many authors (Brown 1963, Lowenstein and Swieca 1971, Coleman 1975, 1976, Coleman *et al* 1975). People often use the axial (also Coulomb) gauge $A_1 = 0$ and find the solution of A_0

$$A_0(x) = -\tfrac{1}{2}e\int j_0(x')|x-x'|\,dx - Fx + \text{constant}, \qquad j_0 = \psi^+\psi. \tag{5.4}$$

One can see that A_0 is entirely fixed by the distribution of matter charge density j_0 besides a contribution from the background field F. Actually, in one-dimensional space, due to the absence of transversal polarisation, there is no independent degree of freedom for the electromagnetic field and also no spin for the fermion field.

The Hamiltonian can be written as

$$H=\int \mathrm{d}x\bar{\psi}(\mathrm{i}\gamma_1\partial_1+m)\psi-\tfrac{1}{4}e^2\int \mathrm{d}x\,\mathrm{d}y\,j_0(x)|x-y|j_0(y)-eF\int \mathrm{d}x\,xj_0(x)+\text{constant}. \tag{5.5}$$

It should be stressed that the fermion version of the Schwinger model suffers from some ambiguity in definitions of currents as $j^\mu=\bar{\psi}\gamma^\mu\psi$ or $j_5^\mu=\bar{\psi}\gamma^\mu\gamma_5\psi$, though they are formally conserved. The non-locality of these currents (Wang 1984) proves an obstacle when dealing with the phase transition problem as divergence emerges in calculations. Fortunately, there is an interesting correspondence between the fermion field and the boson field in one-dimensional space (Casher *et al* 1973, 1974, Kogut and Susskind 1974, 1975, Mandelstam 1975)

$$\begin{aligned} &j^\mu\leftrightarrow(1/\sqrt{\pi})\varepsilon^{\mu\nu}\,\partial_\nu\phi, && j_5^\mu=\varepsilon^{\mu\nu}j_\nu\leftrightarrow(1/\sqrt{\pi})\,\partial^\mu\phi\\ &\mathrm{i}\bar{\psi}\gamma^\mu\,\partial_\mu\psi\leftrightarrow\tfrac{1}{2}\partial^\mu\phi\,\partial_\mu\phi, && \bar{\psi}\psi\leftrightarrow 2K^2\!:\cos 2\sqrt{\pi}\phi: \end{aligned} \tag{5.6}$$

Then the equivalent Hamiltonian for the Schwinger model is

$$\mathscr{H}=N_\mu\left[\frac{1}{2}\pi_\phi^2+\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2+\frac{1}{2}\sigma^2\left(\phi+\frac{\theta}{2\sqrt{\pi}}\right)^2+\frac{\tilde{m}_0^2}{4\pi}\cos(2\sqrt{\pi}\phi)\right] \tag{5.7}$$

where

$$\sigma=e/\sqrt{\pi},\qquad \theta=2\pi F/e. \tag{5.8}$$

N_μ represents the normal order with respect to a boson of mass μ, and the parameter

$$\tilde{m}_0^2=8\pi mK^2=c\mu m \tag{5.9}$$

where $c=2\,\mathrm{e}^\gamma$, $\gamma=0.5772$ being the Euler constant (Wang 1984).

Then in the boson version we can easily perform quantisation of the field and take the average value of the Hamiltonian in the thermal coherent state. The result is

$$\begin{aligned} U&=\int \mathrm{d}x\,{}_\beta\langle f|\mathscr{H}|f\rangle_\beta\\ &=\frac{L}{2\pi}[(I_1+I_2)T^2+\sigma^2I_3]+\frac{1}{2}\int \mathrm{d}x\left(\frac{d\tilde{z}}{\mathrm{d}x}\right)^2+\frac{1}{2}\sigma^2\int \mathrm{d}x\left(\tilde{z}(x)+\frac{\theta}{2\sqrt{\pi}}\right)^2\\ &\quad+\frac{1}{4\pi}\tilde{m}^2\int \mathrm{d}x\cos(2\sqrt{\pi}\tilde{z}(x)) \end{aligned} \tag{5.10}$$

where

$$\tilde{m}^2=\tilde{m}_0^2\,\mathrm{e}^{-2I_3(T)}. \tag{5.11}$$

The function $\tilde{z}(x)$ is defined as the same as in equation (4.6) and obeys the following equation:

$$(\mathrm{d}^2\tilde{z}/\mathrm{d}x^2)+(\tilde{m}^2/2\sqrt{\pi})\sin(2\sqrt{\pi}\tilde{z})-\sigma^2\left(\tilde{z}+\frac{\theta}{2\sqrt{\pi}}\right)=0. \tag{5.12}$$

A space independent solution of $\tilde{z}(x)$ can be found by solving the following equation (figure 1):

$$\sin 2\sqrt{\pi}\tilde{z}=(2\sqrt{\pi}\sigma^2/\tilde{m}^2)(\tilde{z}+\theta/2\sqrt{\pi}). \tag{5.13}$$

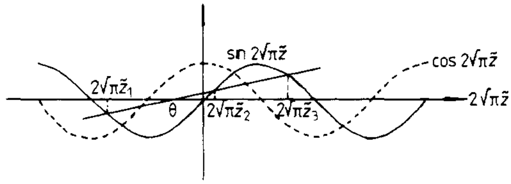

Figure 1.

We shall always choose the solutions of (5.13) to lie in the interval $[-\pi, \pi]$. There are three roots $\tilde{z}_1$, $\tilde{z}_2$ and $\tilde{z}_3$ in general.

The mass square of phonon in a given uniform phase will be evaluated by

$$M^2(T) = \langle V''(\phi)\rangle = -\tilde{m}_0^2\langle\cos 2\sqrt{\pi}\phi\rangle + \sigma^2$$
$$= -\tilde{m}^2 \cos 2\sqrt{\pi}\tilde{z} + \sigma^2. \tag{5.14}$$

It is this M which will be adopted as the quantisation mass, i.e. $\mu = M$. There are three different cases which have to be examined.

(1) *The massless Schwinger model* ($m = 0$)

This is an exactly soluble case at zero temperature. Now the space independent solution of equation (5.12):

$$\tilde{z} = -\theta/2\sqrt{\pi} \tag{5.15}$$

implies that the uniform condensation always screens the background field F. However, the condensation is still in a temperature dependent two-fluid state as shown by equation (4.7).

The temperature independent spectrum of the boson is well known

$$M^2 = \sigma^2 = e^2/\pi \tag{5.16}$$

(2) *The massive Schwinger model in the absence of a background field*
In this case $\theta = 0$. Equation (5.13) becomes

$$\sin 2\sqrt{\pi}\tilde{z} = \frac{\sigma^2}{\tilde{m}^2}(2\sqrt{\pi}\tilde{z}). \tag{5.17}$$

The solutions of (5.17) are $\tilde{z}_2 = 0$, $\tilde{z}_1 = -\tilde{z}_3$. Notice that

$$U(\tilde{z}_1) = U(\tilde{z}_3) < U(0). \tag{5.18}$$

and

$$M^2(T) = -\tilde{m}^2 \cos 2\sqrt{\pi}\tilde{z}_1 + \sigma^2 > 0. \tag{5.19}$$

Combining (5.19) with (5.17), we have

$$\tan 2\sqrt{\pi}\tilde{z} = \frac{\sigma^2}{\sigma^2 - M^2}(2\sqrt{\pi}\tilde{z}). \tag{5.20}$$

Combining (5.19) with (5.20) and noting $\tilde{m}^2 = cMm\,\mathrm{e}^{-2I_3}$, we have

$$M^4 - (2\sigma^2 + c^2m^2\,\mathrm{e}^{-4I_3})M^2 + \sigma^4(1 + 4\pi\tilde{z}^2) = 0. \tag{5.21}$$

We should determine $\tilde{z}$ from (5.17) and substitute it into (5.21) to find M. Two cases will be discussed separately.

(2a) *For a weak coupling Schwinger model*

$$\sigma^2/m^2 \ll 1. \tag{5.22}$$

At low temperature, we have approximately

$$\tilde{z}_3 = \tfrac{1}{2}\sqrt{\pi}(1-\sigma^2/\tilde{m}^2). \tag{5.23}$$

The solution of (5.21) is†

$$M^2 \simeq \tfrac{1}{2}[(2\sigma^2 + c^2m^2\,\mathrm{e}^{-4I_3}) + (c^4m^4\,\mathrm{e}^{-8I_3} + 4\sigma^2c^2m^2\,\mathrm{e}^{-4I_3} - 4\pi^2\sigma^4)^{1/2}] \tag{5.24}$$

$$M^2 \xrightarrow[T\to 0]{} \sigma^2 + c^2m^2 \xrightarrow[\sigma\to 0]{} c^2m^2. \tag{5.25}$$

At the high-temperature limit, $\tilde{m}^2$ decreases to zero, $\tilde{z}\to 0$ in (5.17), then (5.19) in turn gives $M=\sigma$. There is no critical temperature. The condensation tends to zero only at the extremely high-temperature limit.

(2b) *For a strong coupling Schwinger model*

$$\sigma^2/m^2 \gg 1. \tag{5.26}$$

The condition for a real root for M^2 in (5.21) can only be obtained by demanding

$$\tilde{z} = 0, \tag{5.27}$$

thus no condensation at all. In this case

$$M^2 = \tfrac{1}{2}[2\sigma^2 + c^2m^2\,\mathrm{e}^{-4I_3} \pm (c^4m^4\,\mathrm{e}^{-8I_3} + 4\sigma^2c^2m^2\,\mathrm{e}^{-4I_3})^{1/2}]. \tag{5.28}$$

By inspecting (5.10), we choose the lower (minus) sign. Then

$$M_2 \xrightarrow[T\to 0]{} \tfrac{1}{2}(2\sigma^2 + c^2m^2 - 2\sigma cm\,\mathrm{e}^{-2I_3}) \tag{5.29}$$

$$M^2 \xrightarrow[T\to\infty]{} \sigma^2 + \tfrac{1}{2}c^2m^2. \tag{5.30}$$

(3) *The massive Schwinger model in the presence of background field* ($\theta \neq 0$)
Suppose that $0<\theta<\pi$, we need only consider $\tilde{z}_1$ in figure 1 because $U(\tilde{z}_1)$ is the lowest one in these three phases.

(3a) *Weak coupling case* ($\sigma^2 \ll m^2$)
We have at low temperature

$$\tilde{z}_1 = -\tfrac{1}{2}\sqrt{\pi}[1-(\sigma^2/\tilde{m}^2)(1-\theta/\pi)]. \tag{5.31}$$

The phonon spectrum (5.24) is nearly unchanged in this approximation, but as the termperature increases gradually, $\tilde{z}_1$ tends to a limiting value

$$\lim_{T\to\infty} \tilde{z}_1 = -\theta/2\sqrt{\pi}. \tag{5.32}$$

It happens to be precisely that situation in the massless case, since this time we have again $\lim_{T\to\infty}\tilde{m}^2(T)=0$ and

$$\lim_{T\to\infty} M^2(T) = \sigma^2 \tag{5.33}$$

† Another solution of (5.21) has been discarded because it is $M'^2 \sim \pi^2\sigma^4/c^2m^2 \xrightarrow[\alpha\to 0]{} 0$ being unreasonable by inspecting on $V''(\phi)$ directly.

as expected. In the high-temperature limit, the pair creation process will prevail over the suppression mechanism provided by the rest mass of the fermion until the total shielding of the external field is achieved.

(3b) *The strong coupling case* ($\sigma^2 \gg m^2$)
Since the fermion mass can be neglected, it is not surprising that, the stable solution of (5.17) and (5.21) implies that the total shielding condition

$$\tilde{z} = -\theta/2\sqrt{\pi} \tag{5.34}$$

must be realised at the low-temperature limit, let alone the high-temperature range. But the phonon spectrum is the same as that in the $\theta = 0$ case as shown by (5.28).

Next, we consider the interval of θ which lies within $(-\pi, 0)$. Careful inspection of figure 1 reveals the same qualitative results as before. No new feature emerges except the choice of one in three roots to ensure the lowest energy. The property of the Schwinger model will be a periodical function of θ with period 2π.

6. Summary and discussion

We propose a new representation of the two-fluid model in relativistic field theory, the thermal coherent state, in some detail. Whilst one can see that the accuracy of calculation in this formulation is the same as that in the Green function approach to the lowest order (I and II), it does provide the advantage of simplifying the calculation and provide more direct insight into the physical problem.

As the ϕ^4 and sine-Gordon models had been discussed in I and II, in this paper we have put most emphasis on the multisoliton solution. Since the multisoliton configurations are excited within a nonlinear system, they are highly correlated and their locations are fixed by the boundary condition. Being a thermal coherent state, a multisoliton configuration carries two components as its inner structure, one is essentially coherent and remains at zero temperature and the other incoherent excitations attached to the former are related to the temperature. To our understanding, this picture may be different from that of the ideal gas model of the multisoliton system discussed in the literature (Bishop 1981, Maki and Takayama 1979a, b). We hope that further study will clarify this problem and make contact with experimental investigations.

For the Schwinger model, only the space uniform solutions are discussed because we cannot find the exact non-uniform solutions for equation (5.12), but by comparing the discussion on the ϕ^4 or the sine-Gordon system, we expect no substantial change in qualitative behaviour for the Schwinger model even if a non-uniform condensation does occur. Although the property of the Schwinger model at zero temperature has been examined extensively in the literature, few papers are devoted to its behaviour at high temperature. Our results are in conformity with those of Love (1981).

Finally, let us go back to the problem of mass. In this paper we simply resort to the definition of mass M after symmetry breaking as

$$M^2 = \langle V''(\phi_m)\rangle, \qquad \langle V'(\phi_m)\rangle = 0 \tag{6.1}$$

where the average is taken with respect to a thermal coherent state with a quantised ϕ field carrying mass μ which is set to $\mu = M$. Certainly, this is merely a formal manipulation for evading more complicated but more fundamental evaluations as used in the Green function method which is formidable for the multisoliton system.

However, we can get some information about the nature of mass and its relevance to the critical temperature. At a certain temperature, the mass of elementary excitations (quasiparticles) receives two contributions added together. One is coherent in its essential stemming from spontaneously broken symmetry and may be weakly dependent on the temperature. The other is incoherent and strongly dependent on the temperature. Then a critical temperature may exist above which the symmetry is restored. The ϕ^4 model is a typical example of this type. On the other hand, if the mass consists of only one term which is temperature dependent, there will be no critical temperature. It is just this case which occurs in the sine-Gordon model. The Schwinger model is the most complicated case. At zero temperature the boson is actually a bound state whose mass comes from the long-range gauge field coupling, because in one-dimensional space, the coupling constant e carries a mass dimension. The higher the temperature (with dimension of mass), the less important the other mass parameter m will be, so no critical temperature exists. A recent investigation on the Higgs mechanism in gauge theories will provide some other interesting information about mass (Ni 1984).

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Appendix 1. The proof of equation (2.10)

We want to calculate the expectation value of $\hat{a}_p^+\hat{a}_q$ in the thermal coherent state $|f\rangle_\beta$

$$\begin{aligned}
{}_\beta\langle f|\hat{a}_p^+\hat{a}_q|f\rangle_\beta &= N^2{}_\beta\langle 0|\exp\Big(\sum_k f_k^*\hat{a}_k\Big)\hat{a}_p^+\hat{a}_q\exp\Big(\sum_{k'} f_{k'}\hat{a}_{k'}^+\Big)|0\rangle_\beta \\
&= N^2{}_\beta\langle 0|(\hat{a}_p^+ + f_p^*)\exp\Big(\sum_k f_k^*\hat{a}_k\Big)\exp\Big(\sum_{k'} f_{k'}\hat{a}_{k'}^+\Big)(\hat{a}_q+f_q)|0\rangle_\beta \\
&= N^2\exp\Big(\sum_k |f_k|^2\Big){}_\beta\langle 0|(\hat{a}_p^+ + f_p^*)\exp\Big(\sum_{k'} f_{k'}\hat{a}_{k'}^+\Big)\exp\Big(\sum_k f_k^*\hat{a}_k\Big)(\hat{a}_q+f_q)|0\rangle_\beta .
\end{aligned} \tag{A1.1}$$

The contraction between $\hat{a}_p^+$ and $\hat{a}_q$ yields $n_p\delta_{pq}$ whereas the contraction between $\hat{a}_{k'}^+$ and $\hat{a}_k$ yields $n_k\delta_{kk'}$. Notice, however, there are $n!$ possibilities of contractions between $\hat{a}_{k'}^+$ and $\hat{a}_k$ in each term containing $(\hat{a}_{k'}^+)^n(\hat{a}_k)^n$. Therefore

$$\begin{aligned}
&{}_\beta\langle 0|\bigg[\sum_m \frac{1}{m!}\Big(\sum_{k'} f_{k'}\hat{a}_{k'}^+\Big)^m\bigg]\bigg[\sum_n \frac{1}{n!}\Big(\sum_k f_k^*\hat{a}_k\Big)^n\bigg]|0\rangle_\beta \\
&\qquad = \sum_{m,n}\frac{\delta_{nm}}{n!m!}\Big(\sum_{kk'} f_{k'}f_k^* n_k\delta_{kk'}\Big)^n(n!) = \exp\Big(\sum_k |f_k^2|n_k\Big).
\end{aligned} \tag{A1.2}$$

Furthermore, the contraction between $\hat{a}_p^+$ and a_k must be accompanied by the contraction between $\hat{a}_{k'}^+$ and a_q, which results in $n_pn_qf_p^*f_q$. Collecting every term together, we prove formula (2.10):

$$ {}_\beta\langle f|\hat{a}_p^+\hat{a}_q|f\rangle_\beta = \delta_{pq}n_p + f_p^*f_q(1+n_p)(1+n_q) \tag{A1.3}$$

where equation (2.5) has been used.

Appendix 2. The proof of equation (2.15)

First, we expand $\mathrm{e}^{\mathrm{i}g\phi}$ in normal order

$$:\mathrm{e}^{\mathrm{i}g\phi}: = \exp\left(\mathrm{i}g\sum_k (2\omega_k L)^{-1/2}\,\mathrm{e}^{-\mathrm{i}kx}\,\hat{a}_k^+\right)\exp\left(\mathrm{i}g\sum_k (2\omega_k L)^{-1/2}\,\mathrm{e}^{\mathrm{i}kx}\,a_k\right)$$
$$= \sum_{n,m=0}^{\infty}\frac{1}{n!m!}\left(\mathrm{i}g\sum_p (2\omega_p L)^{-1/2}\,\mathrm{e}^{-\mathrm{i}px}\,\hat{a}_p^+\right)^m$$
$$\times\left(\mathrm{i}g\sum_q (2\omega_q L)^{-1/2}\,\mathrm{e}^{\mathrm{i}px}\,\hat{a}_q\right)^n. \tag{A2.1}$$

Then

$$_\beta\langle f|:\mathrm{e}^{\mathrm{i}g\phi}:|f\rangle_\beta = 1+\sum_{n=1}^{\infty}\frac{1}{n!}G^n+\sum_{m=1}^{\infty}\frac{1}{m!}(-G^*)^m+F, \tag{A2.2}$$

where

$$G^n = \sum_{q_1\dots q_n}\prod_{j=1}^{n}[\mathrm{i}g(2\omega_{q_j}L)^{-1/2}\,\mathrm{e}^{\mathrm{i}q_jx}]_\beta\langle f|\hat{a}_{q_1}\dots\hat{a}_{q_n}|f\rangle_\beta$$
$$= \sum_{q_1\dots q_n}\prod_{j=1}^{n}[\mathrm{i}g(2\omega_{q_j}L)^{-1/2}\,\mathrm{e}^{\mathrm{i}q_jx}f_{q_j}(1+n_{q_j})]$$
$$= \left\{\sum_q[\mathrm{i}g(2\omega_q L)^{-1/2}\,\mathrm{e}^{\mathrm{i}qx}f_q(1+n_q)]\right\}^n, \tag{A2.3}$$

$$F = \sum_{m=1}^{\infty}\frac{1}{m!}\left[\sum_{p_1\dots p_m}\prod_{i=1}^{m}[\mathrm{i}g(2\omega_{p_i}L)^{-1/2}\,\mathrm{e}^{-\mathrm{i}p_ix}]\left(\sum_{q_1}[\mathrm{i}g(2\omega_{q_1}L)^{-1/2}\,\mathrm{e}^{\mathrm{i}q_1x}]_\beta\langle f|\hat{a}_{p_1}^+\dots\hat{a}_{p_m}^+\hat{a}_{q_1}|f\rangle_\beta\right.\right.$$
$$+\frac{1}{2!}\sum_{q_1q_2}[\mathrm{i}g(2\omega_{q_1}L)^{-1/2}\,\mathrm{e}^{\mathrm{i}q_1x}][\mathrm{i}g(2\omega_{q_2}L)^{-1/2}\,\mathrm{e}^{\mathrm{i}q_2x}]$$
$$\left.\left.\times_\beta\langle f|\hat{a}_{p_1}^+\dots\hat{a}_{p_m}^+\hat{a}_{q_1}\hat{a}_{q_2}|f\rangle_\beta+\dots\right)\right]. \tag{A2.4}$$

Noticing the formula (2.14) and the definitions of (2.17) and (2.18), we get

$$F = \sum_{m=1}^{\infty}\frac{1}{m!}\Big([(-G^*)^mG+C_1^m(-G^*)^{m-1}(-K)]$$
$$+\frac{1}{2!}[(-G^*)^mG^2+C_1^mC_1^2(-G^*)^{m-1}(-K)G$$
$$+C_2^mC_2^2(2!)(-G^*)^{m-2}(-K)^2]$$
$$+\dots+\frac{1}{(m-1)!}[(-G^*)^mG^{m-1}+\dots$$
$$+C_{m-1}^mC_{m-1}^{m-1}(m-1)!(-G^*)(-K)^{m-1}]$$
$$+\frac{1}{m!}[(-G^*)^mG^m+\dots+C_{m-1}^mC_{m-1}^m(m-1)!(-G^*)(-K)^{m-1}G$$
$$+m!(-K)^m]$$
$$+\frac{1}{(m+1)!}[(-G^*)^mG^{m+1}+\dots+C_m^mC_m^{m+1}m!G(-K)^m]+\dots\Big) \tag{A2.5}$$

where $C_r^n = n!/(n-r)!r!$ is the combination number. After some algebra, we get

$$F = \sum_{m=1}^{\infty} \frac{1}{m!}\{[e^G(-K-G^*)^m] - (-G^*)^m\}$$
$$= e^G(e^{-K-G^*}-1) - e^{-G^*}+1. \tag{A2.6}$$

The substitution of (A2.6) into (A2.2) completes the proof of equation (2.15):

$${}_\beta\langle f|:e^{ig\phi}:|f\rangle_\beta = \exp(-K+G-G^*). \tag{A2.7}$$

Appendix 3. Some mathematical formulae

$$I_1 = \int_0^\infty dx\, x^2(x^2+a^2)^{-1/2}[\exp(x^2+a^2)^{1/2}-1]^{-1} = a\sum_{n=1}^{\infty}\frac{K_1(na)}{n} \tag{A3.1}$$

$$I_1(a) = (\pi a/2)^{1/2} e^{-a}[1+O(1/a)] \qquad (a \gg 1)$$

$$I_1(a) = \tfrac{1}{6}\pi^2 - \tfrac{1}{2}\pi a - \tfrac{1}{4}a^2[\ln(a/4\pi)+\gamma-\tfrac{1}{2}] + O(a^4) \qquad (a \ll 1)$$

$$I_2 = \int_0^\infty dx\,(x^2+a^2)^{1/2}[\exp(x^2+a^2)^{1/2}-1]^{-1} = \frac{a^2}{2}\sum_{n=1}^{\infty}[K_2(na)+K_0(na)] \tag{A3.2}$$

$$I_2(a) = a(\pi a/2)^{1/2} e^{-a}[1+O(1/a)] \qquad (a \gg 1)$$

$$I_2(a) = \tfrac{1}{6}\pi^2 + \tfrac{1}{4}a^2[\ln(a/4\pi)+\gamma+\tfrac{1}{2}] + O(a^4) \qquad (a \ll 1)$$

$$I_3 = \int_0^\infty dx\,(x^2+a^2)^{-1/2}\exp(x^2+a^2)^{1/2}-1)^{-1} = \sum_{n=1}^{\infty}K_0(na) \tag{A3.3}$$

$$I_3(a) = (\pi/2a)^{1/2} e^{-a}[1-1/8a+O(1/a^2)] \qquad (a \gg 1)$$

$$I_3(a) = \frac{\pi}{2a} + \tfrac{1}{2}\ln\frac{a}{4\pi} + \tfrac{1}{2}\gamma - \frac{\pi\xi(3)}{2(2\pi)^3}a^2 + O(a^4) \qquad (a \ll 1)$$

where $\gamma = 0.577\ldots$; $\xi(z) = 1.20\ldots$; K_0, K_1 and K_2 are Bessel functions of the imaginary argument.

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